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THE TRANSMUTED GEOMETRIC-G FAMILY OF DISTRIBUTIONS: THEORY AND APPLICATIONS

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ABSTRACT

We introduce a new family of continuous distributions called the transmuted geometric-G family which extends the transmuted family pioneered by Shaw and Buckley (2007). Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Rényi and Shannon entropies, order statistics and probability weighted moments are derived. Some special models of the new family are provided. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the proposed family are illustrated by two applications to real data sets.

KEY WORDS

Transmuted-G Class, Maximum Likelihood, Generating Function, Order Statistic, Probability Weighted Moments.

1. INTRODUCTION

Recently, several generalized families of continuous distributions have been proposed and applied to model various phenomena. However, there is a clear need for extended forms of the well-known distributions by adding one or more shape parameter(s) in order to obtain greater flexibility in modelling various data.

Some well-known families are the Marshall-Olkin-G (MO-G) by Marshall and Olkin (1997), the beta-G (B-G) by Eugene et al. (2002), the transmuted-G (T-G) by Shaw and Buckley (2007), the Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011), the McDonald-G (Mc-G) by Alexander et al. (2012), the gamma-G by Zografos and Balakrishanan (2009), the Kumaraswamy odd log-logistic-G (KwOLL-G) by Alizadeh et al. (2015), the beta odd log-logistic generalized by Cordeiro et al. (2015), the generalized transmuted-G (GT-G) by Nofal et al. (2015), the transmuted exponentiated generalized-G (TExG-G) by Yousof et al. (2015) and the Kumaraswamy transmuted-G family (Kw-TG) by Afify et al. (2016).

Let $p(t)$ be the probability density function (pdf) of a random variable $T \in [a,b]$ for $-\infty < a < b < \infty$ and let $W[G(x)]$ be a function of the cumulative distribution function (cdf) of a random variable X such that $W[G(x)]$ satisfies the following conditions:

$$\begin{aligned}&(\mathrm{i}) \quad W\left[G(x)\right] \in \left[a,b\right],\\ &(\mathrm{ii}) \quad W\left[G(x)\right] \text{ is differentiable and monotonically nondecreasing, and}\\ &(\mathrm{iii}) \quad W\left[G(x)\right] \to a \;\; as \;\; x \to -\infty \text{ and } W\left[G(x)\right] \to b \;\; as \;\; x \to \infty.\end{aligned} \tag{1}$$

Recently, Alzaatreh et al. (2013) defined the T-X family of distributions by

$$F(x) = \int_a^{W[G(x)]} p(t)\,dt, \tag{2}$$

where W[G(x)] satisfies conditions (1). The pdf corresponding to (2) is given by

$$f(x) = \left\{\frac{d}{dx} W\left[G(x)\right]\right\} p\left\{W\left[G(x)\right]\right\}. \tag{3}$$

The objective of this study is to define a new family of distributions called the transmuted geometric-G (TG-G for short) family of distributions and study its mathematical properties.

Based on the T-X family, we construct a new generator by taking $W\left[G(x)\right] = \frac{\theta G(x)}{1+(\theta-1)G(x)}$ and $p(t) = 1+\lambda-2\lambda t$, $0 < t < 1$. Then, the CDF of the TG-G family is given by

$$F(x) = \int_0^{\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}} (1+\lambda-2\lambda t)\,dt = \frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\frac{\lambda \bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right], \tag{4}$$

where $G(x;\phi)$ is the baseline CDF and $\theta > 0$ and $|\lambda| \le 1$ are two additional shape parameters. The TG-G is a wider class of continuous distributions. It includes the transmuted-G family of distributions and geometric-G family.

The rest of the paper is organized as follows. In Section 2, we define the TG-G family. A useful mixture representation for the pdf of the new family is derived in Section 3. In Section 4, we present two special models and plots of their pdf's and hrf's. In Section 5, we derive some of its general mathematical properties including quantile function, ordinary and incomplete moments, mean deviations, moment generating function (mgf), Rényi, Shannon and q-entropies. Order statistics and their moments are investigated in Section 6. In Section 7, we obtain the probability weighted moments

(PWMs) of the proposed family. Maximum likelihood estimation (MLE) of the model parameters is addressed in Section 8. In Section 9, we provide two applications to real data to illustrate the importance and flexibility of the new family. Finally, some concluding remarks are presented in Section 10.

2. THE TG-G FAMILY

The pdf corresponding of (4) is given by

$$f(x)=\frac{\theta g(x;\phi)}{\left[1+(\theta-1)G(x;\phi)\right]^{2}}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]. \tag{5}$$

For $\lambda = 0$ we obtain geometric-G (GG) family. We denote by $X\sim$TG-G(λ,θ,ϕ) a random variable having density function (5). The reliability function ($R(x)$) and hrf ($\tau(x)$) of X are, respectively, given by

$$R(x)=1-\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\frac{\lambda\bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$$

and

$$\tau(x)=\frac{\dfrac{\theta g(x;\phi)}{\left[1+(\theta-1)G(x;\phi)\right]^{2}}\left[1+\lambda-\dfrac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]}{1-\dfrac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\dfrac{\lambda\bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]}.$$

Below is a simple motivation for the development of TG-G family of distributions. Suppose $Z_1$and Z_2 be two random variables from $\theta G(x;\phi)/\left[1+(\theta-1)G(x;\phi)\right]$. Define

$$X=\begin{cases} Z_{1:2} \text{ with probability } \dfrac{1+\lambda}{2}; \\ Z_{2:2} \text{ with probability } \dfrac{1-\lambda}{2}, \end{cases}$$

where $Z_{1:2}=\min(Z_1,Z_2)$ and $Z_{2:2}=\max(Z_1,Z_2)$. Then the cdf of X is given by (4).

The TG-G family of distribution appears to be more felxible and could be used for modeling various types of data. For illustration propose we provide pdf and hrf of some special models of this family in figures 1 and 2. It can be seen that the hazard rate can take constant, increasing, decreasing, upside down and bathtub shaped. Therefore, this family of distribution could be used to model diverse nature of data sets.

Henceforth, we will omit the dependence on the model parameters and write simply $g(x)=g(x;\phi)$ and $G(x)=G(x;\phi)$.

3. MIXTURE REPRESENTATION

In this section, we provide a very useful representation for the TG-G density which can be used to study its mathematical characteristics. The pdf (5) can be rewritten as

$$f(x)=\frac{\theta(1+\lambda)\ g(x)}{\left[1+(\theta-1)G(x)\right]^2}-\frac{2\lambda\theta^2\ g(x)\ G(x)}{\left[1+(\theta-1)G(x)\right]^3}. \tag{6}$$

Then, the pdf (6) can be rewritten as

$$f(x)=\left[(1+\lambda)\theta\ g(x)\sum_{k=0}^{\infty}(\theta-1)^k\binom{-2}{k}G^k(x)\right] -\left[2\ \lambda\theta^2 g(x)\sum_{k=0}^{\infty}(\theta-1)^k\binom{-3}{k}G^{k+1}(x)\right]. \tag{7}$$

The pdf (7) can be expressed as a mixture of exp-G densities

$$f(x)=\sum_{k=0}^{\infty}\left[a_k\ \pi_{k+1}(x)-b_k\ \pi_{k+2}(x)\right]. \tag{8}$$

But

$$\binom{-2}{k}=(-1)^k\,(k+1) \quad \text{and} \quad \binom{-3}{k}=\frac{(-1)^k\,(k+1)(k+2)}{2},$$

where $\pi_\alpha(x)=\alpha g(x)G^{(\alpha-1)}(x)$ is the exp-G pdf with power parameter $\alpha>0$,

$$a_k=\theta(1+\lambda)(1-\theta)^k \quad \text{and} \quad b_k=\lambda\ \theta^2\,(k+1)(1-\theta)^k.$$

Thus, several mathematical properties of the TG-G family can be obtained simply from those properties of the exp-G family. Equation (8) is the main result of this section.

The cdf of the TG-G family can also be expressed as a mixture of exp-G densities. By integrating (8), we obtain the same mixture representation

$$F(x)=\sum_{k=0}^{\infty}\left[a_k\ \Pi_{k+1}(x)-\ b_k\ \Pi_{k+2}(x)\right],$$

where $\Pi_\delta(x)$ is the cdf of the exp-G family with power parameter δ.

4. SPECIAL MODELS

In this section, we provide two special models of the TG-G family correspond to the baseline Weibull and Burr X distributions. These special models generalize some well-known distributions in the literature.

4.1 The TG-Weibull (TGW) Distribution

The Weibull distribution with positive parameters α and β has cdf and pdf (for $x > 0$) given by $G(x) = 1 - e^{-(\alpha x)^{\beta}}$ and $g(x) = \beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}$, respectively. Then, the pdf of the TGW model is given by

$$f(x) = \frac{\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}}{\left[1+(\theta-1)\left(1-e^{-(\alpha x)^{\beta}}\right)\right]^{2}}\left[1+\lambda-\frac{2\lambda\,\theta\left(1-e^{-(\alpha x)^{\beta}}\right)}{1+(\theta-1)\left(1-e^{-(\alpha x)^{\beta}}\right)}\right],$$

where α, β and θ are positive parameters and $|\lambda| \leq 1$.

The TGW distribution includes the transmuted Weibull (TW) distribution introduced by Aryal and Tsokos (2011) when $\theta = 1$. The plots of the pdf and hrf of the TGW distribution are displayed in Figure 1 for selected parameter values.

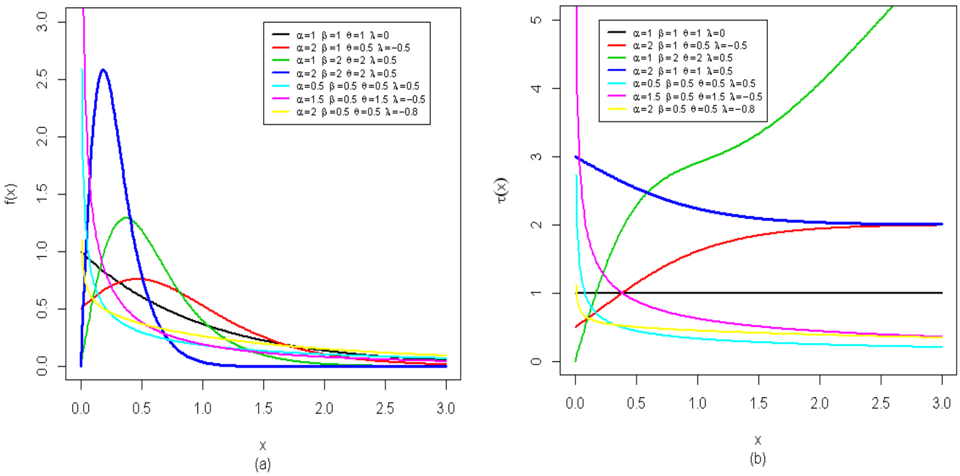

Figure 1: (a) pdf of TGW Distribution (b) hrf of TGW Distribution

4.2 The TG-Burr X (TGBrX) Distribution

The Burr X (also known as the generalized Raleigh) model with positive parameters α and β has cdf and pdf (for $x > 0$) given by $G(x) = [1 - e^{-(\beta x)^2}]^{\alpha}$ and $g(x) = 2\alpha\beta^2 x e^{-(\beta x)^2}[1 - e^{-(\beta x)^2}]^{\alpha-1}$ respectively. Then, the TGBrX density reduces to

$$f(x) = \frac{2\alpha\theta\beta^{2}xe^{-(\beta x)^{2}}\left(1-e^{-(\beta x)^{2}}\right)^{\alpha-1}}{\left[1+(\theta-1)\left(1-e^{-(\beta x)^{2}}\right)^{\alpha}\right]^{2}}\left[1+\lambda-\frac{2\lambda\,\theta\left(1-e^{-(\beta x)^{2}}\right)^{\alpha}}{1+(\theta-1)\left(1-e^{-(\beta x)^{2}}\right)^{\alpha}}\right],$$

where α, β and θ are positive parameters and $|\lambda| \leq 1$.

Some plots of the pdf and hrf of the TGBrX distribution are given in Figure 2 for selected parameter values.

5. MATHEMATICAL PROPERTIES

In this section, we provide some mathematical properties of the TG-G family including quantile function (qf), moments, generating function, incomplete moments, residual and reversed residual lifes and entropies.

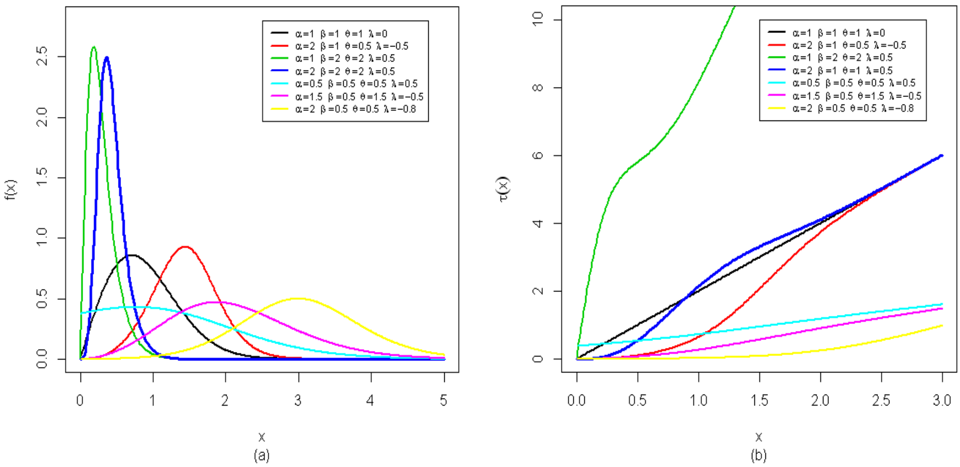

Figure 2: (a) pdf of TGBrX Distribution (b) hrf of TGBrX Distribution

5.1 Quantile Function

The quantile function (qf) of X, where $X \sim \text{TG-G}(\lambda, \theta, \phi)$, is obtained by inverting (4). The qf, $Q(u)$, of X is given by

$$Q(u) = F^{(-1)}(u) = G^{(-1)}\left\{\frac{\lambda+1-\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda\theta+(1-\theta)\left[\lambda+1-\sqrt{(\lambda+1)^2-4\lambda u}\right]}\right\},\ 0<u<1.$$

for $\lambda \neq 0$. For $\lambda = 0$, we have

$$Q(u) = G^{-1}\left[\frac{u}{\theta+(1-\theta)u}\right].$$

Simulating the TG-G random variable is straightforward. If U is a uniform variate on the unit interval$(0,1)$, then the random variable $X = Q(U)$ follows the TG-G distribution.

5.2 Moments

Henceforth, Y_k denotes the exp-G distribution with power parameter k. The rth moment of X, say μ'_r, follows from (9) as

$$\mu'_r = E\left(X^r\right) = \sum_{k=0}^{\infty}\left[a_k\ E\left(Y_{k+1}^r\right) - b_k\ E\left(Y_{k+2}^r\right)\right].$$

The nth central moment of X, say M_n, is given by

$$M_n = E\left(X - \mu'_1\right)^n = \sum_{r=0}^{n}\binom{n}{r}\left(-\mu'_1\right)^{(n-r)} E\left(X^r\right)$$
$$= \sum_{r=0}^{n}\sum_{k=0}^{\infty}(-1)^{n-r}\binom{n}{r}\mu_r'^{(n-r)}\left[a_k E\left(Y_{(k+1)}^r\right) - b_k E\left(Y_{(k+2)}^r\right)\right].$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1}\binom{n-1}{r-1}\kappa_r\ \mu'_{n-r},$$

where $\kappa_1 = \mu'_1$, $\kappa_2 = \mu'_2 - \mu_1'^2, \kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu_1'^3$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

5.3 Generating Function

Here, we provide two formulae for the mgf $M_X(t) = E(e^{t\,X})$ of X. Clearly, the first one can be derived from equation (8) as

$$M_X\left(t\right) = \sum_{k=0}^{\infty}\left[a_k\ M_{k+1}\left(t\right) - b_k\ M_{k+2}\left(t\right)\right],$$

where $M_k(t)$ is the mgf of Y_k. Hence, $M_X(t)$ can be determined from the exp-G generating function.

A second formula for $M_X(t)$ follows from (8) as

$$M_X\left(t\right) = \sum_{k=0}^{\infty}\left[a_k\ \tau\left(t,k\right) - b_k\ \tau\left(t,k+1\right)\right],$$

where $\tau(t,k) = \int_0^1 \exp[t\ Q_G(u)]\ u^k du$ and $Q_G(u)$ is the qf corresponding to $G(x)$, i.e., $Q_G(u) = G^{-1}(u)$.

5.4 Incomplete Moments

The sth incomplete moment, say $\varphi_s(t)$, of X can be expressed from (8) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{k=0}^{\infty}[a_k \int_{-\infty}^{t} x^s \pi_{(k+1)}(x)dx - b_k \int_{-\infty}^{t} x^s \pi_{(k+2)}(x)dx]. \quad (9)$$

The mean deviations about the mean $[\theta_1 = E(|X - \mu'_1|)]$ and about the median $[\theta_2 = E\ (|\ X - M\ |)]$ of X are given by $\theta_1 = 2\,\mu_1'\ F(\mu'_1) - 2\varphi_1\ (\mu'_1)$ and $\theta_2 = \mu'_1 - 2\varphi_1(M)$, respectively, where $\mu'_1 = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu'_1)$ is easily calculated from (4) and $\varphi_1(t)$ is the first incomplete moment given by (9) with $s = 1$.

Now, we provide two ways to determine θ_1 and θ_2. First, a general equation for $\varphi_1(t)$ can be derived from (9) as

$$\varphi_1(t) = \sum_{k=0}^{\infty} \left[a_k \; J_{k+1}(x) - \; b_k \; J_{k+2}(x) \right],$$

where $J_k(x) = \int_{-\infty}^{t} x \, \pi_k(x) dx$ is the first incomplete moment of the exp-G distribution.

A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k=0}^{\infty} \left[a_k \; \upsilon_k(t) - \; b_k \; \upsilon_{k+1}(t) \right],$$

where $\upsilon_k(t) = (k+1) \int_0^{G(t)} Q_G(u) \, u^k du$ can be computed numerically.

These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q)/(\pi \mu_1')$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π.

5.5 Residual and Reversed Residual Life Functions

The nth moment of the residual life, say $m_n(t) = E[(X-t)^n | X > t]$, $n = 1,2,...$, is given by

$$m_n(t) = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n dF(x).$$

Therefore,

$$m_n(t) = \frac{1}{R(t)} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r} \sum_{k=0}^{\infty} \left[a_k \int_t^{\infty} x^r \pi_{k+1} - b_k \int_t^{\infty} x^r \; \pi_{k+2}(x) \right].$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t)|X > t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in the last equation.

The nth moment of the reversed residual life, say $M_n(t) = E[(t-X)^n | X \leq t]$ for $t > 0$ and $n = 1,2,...$, is defined by

$$M_n(t) = \frac{1}{F(t)} \int_0^{t} (t-x)^n dF(x).$$

Therefore, the nth moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r} \sum_{k=0}^{\infty} \left[a_k \int_0^{t} x^r \pi_{k+1} - b_k \int_0^{t} x^r \pi_{k+2}(x) \right].$$

The mean inactivity time (MIT) is defined by $M_1(t) = E[(t-X)|X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of X can be obtained easily by setting $n=1$ in the above equation.

5.6 Entropies

The Rényi entropy is defined by

$$I_{\delta}(X)=(1-\delta)^{-1}\log\left[\int_{-\infty}^{\infty} f^{\delta}(x)dx\right],\ \delta>0 \text{ and } \delta\neq 1.$$

Using the pdf (6), we can write

$$f^{\delta}(x)=\frac{\theta^{\delta}g^{\delta}(x)(1+\lambda)^{\delta}}{\left[1+(\theta-1)G(x)\right]^{2\delta}}\left\{1-\frac{2\lambda\theta G(x)}{(1+\lambda)\left[1+(\theta-1)G(x)\right]}\right\}^{\delta}.$$

The Taylor series z^{β} is defined as

$$z^{\beta}=\sum_{k=0}^{\infty}(\beta)_k\frac{(z-1)^k}{k!},$$

where k is a positive integer and $(\beta)_k=\beta(\beta-1)\ldots(\beta-k+1)$ is the descending factorial.

Consider $A=\left\{1-\frac{2\lambda\theta G(x)}{(1+\lambda)\left[1+(\theta-1)G(x)\right]}\right\}^{\delta}$.

Applying the last power series to the quantity A, we obtain

$$f^{\delta}(x)=\sum_{i=0}^{\infty}(\delta)_i\frac{(-1)^i 2^i\lambda^i\theta^{(\delta+i)}(1+\lambda)^{\delta-i}}{i!\left[1+(\theta-1)G(x)\right]^{2\delta+i}}g^{\delta}(x)G^{i}(x).$$

Then we can write

$$f^{\delta}(x)=\sum_{k=0}^{\infty}m_k g^{\delta}(x)G^{k+i}(x),$$

where

$$m_k=\sum_{i=0}^{\infty}(\delta)_i\frac{(-1)^i 2^i\lambda^i(1-\theta)^k\theta^{\delta+i}(1+\lambda)^{\delta-i}}{k!\,i!\,\Gamma(2\delta+i)}\Gamma(2\delta+i+k).$$

Then, the Rényi entropy of the TG-G family is given by

$$I_{\delta}(X)=(1-\delta)^{-1}\log\left[\sum_{k=0}^{\infty}m_k\int_{-\infty}^{\infty}g^{\delta}(x)G^{k+i}(x)dx\right].$$

The δ-entropy, say $H_{\delta}(X)$, can be obtained (for $\delta>0$, $\delta\neq 1$) as

$$H_{\delta}(X) = (\delta - 1)^{-1} \log\left\{1 - \left[\int_{-\infty}^{\infty} f^{\delta}(x)\,dx\right]\right\},$$

which follows from the last equation.

The Shannon entropy of a random variable X, say SI, is a special case of the Rényi entropy when $\delta \uparrow 1$ and it is defined by

$$SI = E\left\{-\left[\log f(X)\right]\right\},$$

which follows by taking the limit of $I_{\delta}(X)$ as δ tends to 1.

6. ORDER STATISTICS

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the TG-G family of distributions. The pdf of ith order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^{j} \binom{n-i}{j} F^{\,j+i-1}(x). \tag{10}$$

Then

$$F^{\,j+i-1}(x) = \frac{\theta^{j+i-1} G^{j+i-1}(x)}{\left[1 + (\theta - 1)G(x)\right]^{j+i-1}} \left[1 + \frac{\lambda \bar{G}(x)}{1 + (\theta - 1)G(x)}\right]^{j+i-1}$$

$$= \sum_{w=0}^{\infty} (j+i-1)_{w} \frac{\lambda^{w} \theta^{j+i-1} \left[1 - G(x)\right]^{w} G^{j+i-1}(x)}{w!\left[1 + (\theta - 1)G(x)\right]^{j+i+w-1}}. \tag{11}$$

Using Equations (5) and (11) we get

$$f(x) F^{\,j+i-1}(x) = \sum_{w=0}^{\infty} (j+i-1)_{w} \frac{(1+\lambda)\lambda^{w} \theta^{j+i} g(x)\left[1 - G(x)\right]^{w} G^{j+i-1}(x)}{w!\left[1 + (\theta - 1)G(x)\right]^{j+i+w+1}}$$

$$- \sum_{w=0}^{\infty} (j+i-1)_{w} \frac{2\lambda^{w+1} \theta^{j+i+1} g(x)\left[1 - G(x)\right]^{w} G^{j+i}(x)}{w!\left[1 + (\theta - 1)G(x)\right]^{j+i+w+2}}.$$

Then

$$f(x) F^{\,j+i-1}(x) = \sum_{k=0}^{\infty} \left[\Upsilon_{k} \pi_{k+j+i+m}(x) - \Psi_{k} \pi_{k+j+i+m+1}(x)\right]. \tag{12}$$

Substituting Equation (12) in Equation (10), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j}}{\mathrm{B}(i,n-i+1)}\binom{n-i}{j}\left[\Upsilon_{k}\pi_{k+j+i+m}(x)-\Psi_{k}\pi_{k+j+i+m+1}(x)\right],$$

where

$$\Upsilon_{k}=\sum_{m,w=0}^{\infty}(j+i-1)_{w}\frac{(-1)^{k}(1+\lambda)\lambda^{w}\theta^{j+i}(1-\theta)^{m}\Gamma(w+1)\Gamma(j+i+w+m+1)}{w!m!k!\Gamma(w-k+1)\Gamma(j+i+w+1)[k+j+i+m]},$$

$$\Psi_{k}=\sum_{m,w=0}^{\infty}(j+i-1)_{w}\frac{(-1)^{k}2\lambda^{w+1}\theta^{j+i+1}(1-\theta)^{m}\Gamma(w+1)\Gamma(j+i+w+m+2)}{w!m!k!\Gamma(w-k+1)\Gamma(j+i+w+2)[k+j+i+m+1]}$$

and $\pi_k(x)$ is the exp-G density with power parameter k.

Then, the density function of the TG-G order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those of Y_{a+k}. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{(i:n)}^{q}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j}}{\mathrm{B}(i,n-i+1)}\binom{n-i}{j}\left[\Upsilon_{k}E\left(Y_{(k+j+i+m)}^{q}\right)-\Psi_{k}E\left(Y_{(k+j+i+m+1)}^{q}\right)\right]. \tag{13}$$

Based upon the moments in Equation (13), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TG-G order statistics. They are linear functions of expected order statistics defined by

$$\lambda_{r}=\frac{1}{r}\sum_{d=0}^{r-1}(-1)^{d}\binom{r-1}{d}E\left(X_{r-d:r}\right),\quad r\geq 1.$$

7. PROBABILITY WEIGHTED MOMENTS

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist.

The (s,r)th PWM of X following the TG-G distribution, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r}=E\left[X^{s}F^{r}(x)\right]=\int_{-\infty}^{\infty}x^{s}\,F^{r}(x)\,f(x)dx.$$

$$F^{r}(x)=\frac{\theta^{r}G^{r}(x)}{\left[1+(\theta-1)G(x)\right]^{r}}\left[1+\frac{\lambda\bar{G}(x)}{1+(\theta-1)G(x)}\right]^{r}\sum_{w=0}^{\infty}(r)_{w}\frac{\lambda^{w}\theta^{r}\left[1-G(x)\right]^{w}G^{r}(x)}{w!\left[1+(\theta-1)G(x)\right]^{r+w}}. \tag{14}$$

From Equation (5) and the last equation , we can write

$$f(x)F^r(x) = \sum_{k=0}^{\infty}\left[\Upsilon_k^*\pi_{k+r+m+1}(x) - \Psi_k^*\pi_{k+r+m+2}(x)\right],$$

where

$$\Upsilon_k^* = \sum_{m,w=0}^{\infty}(r)_w \frac{(-1)^k(1+\lambda)\lambda^w\theta^{r+1}(1-\theta)^m\,\Gamma(w+1)\Gamma(r+w+m+2)}{w!m!k!\Gamma(w-k+1)\Gamma(r+w+2)[k+r+m+1]}$$

and

$$\Psi_k^* = \sum_{m,w=0}^{\infty}(r)_w \frac{(-1)^k\,2\lambda^{w+1}\theta^{r+2}(1-\theta)^m\,\Gamma(w+1)\Gamma(r+w+m+3)}{w!m!k!\Gamma(w-k+1)\Gamma(r+w+3)[k+r+m+2]}.$$

Finally, the (s,r)th PWM of X can be obtained from an infinite linear combination of exp-G moments given by

$$\rho_{s,r} = \sum_{k=0}^{\infty}\left[\Upsilon_k^* E\left(Y_{k+r+m+1}^r\right) - \Psi_k^* E\left(Y_{k+r+m+2}^r\right)\right].$$

8. MAXIMUM LIKELIHOOD ESTIMATION

Here, we determine the maximum likelihood estimators (MLEs) of the parameters of the TG-G family of distributions from complete samples only. Let $x_1, \ldots, x_n$ be a random sample from this family with parameter vector $\boldsymbol{\varphi}$, where $\boldsymbol{\varphi} = (\lambda, \theta, \phi^{\mathrm{T}})^{\mathrm{T}}$.

Then, the log-likelihood function for $\boldsymbol{\varphi}$, say $\ell = \ell(\boldsymbol{\varphi})$, is given by

$$\ell = n\log(\theta) + \sum_{i=0}^{n}\log\left[g(x;\phi)\right] - 2\sum_{i=0}^{n}\log(s_i) + \sum_{i=0}^{n}\log(p_i),$$

where

$$s_i = \left[1+(\theta-1)G(x;\phi)\right] \text{ and } p_i = \left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right].$$

The score vector components, say $\mathbf{U}(\boldsymbol{\varphi}) = \frac{\partial\ell}{\partial\boldsymbol{\varphi}} = \left(\frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\phi_k}\right)^T = \left(U_\lambda, U_\theta, U_{\phi_k}\right)^{\mathrm{T}}$, are given by

$$U_\lambda = \sum_{i=0}^{n}\frac{1}{p_i}\left[1-\frac{2\theta G(x;\phi)}{s_i}\right],$$

$$U_\theta = \frac{n}{\theta} - 2\sum_{i=0}^{n}\frac{G(x;\phi)}{s_i} - 2\lambda\sum_{i=0}^{n}\frac{\left[G(x;\phi)s_i - \theta G(x;\phi)^2\right]}{p_i s_i^2}$$

and

$$U_{\phi_k} = \sum_{i=0}^{n} \frac{g'(x_i;\phi)}{g(x;\phi)} - 2\sum_{i=0}^{n} \frac{(\theta-1)G'(x_i;\phi)}{s_i} - 2\lambda\theta \sum_{i=0}^{n} \frac{G'(x_i;\phi)s_i}{p_i s_i^2}$$
$$+2\lambda\theta(\theta-1)\sum_{i=0}^{n} \frac{G(x;\phi)G'(x_i;\phi)}{p_i s_i^2},$$

where $g'(x_i;\phi) = \partial g(x_i;\phi)/\partial\phi_k$ and $G'(x_i;\phi) = \partial G(x_i;\phi)/\partial\phi_k$.

Setting the nonlinear system of equations $U_\lambda = U_\theta = U_{\phi_k} = 0$ and solving them simultaneously yields the MLE $\hat{\boldsymbol{\varphi}} = (\hat{\lambda}, \hat{\theta}, \hat{\phi}^{\mathrm{T}})^{\mathrm{T}}$ of $\boldsymbol{\varphi} = (\lambda, \theta, \phi^{\mathrm{T}})^{\mathrm{T}}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms. For interval estimation of the model parameters, we require the observed information matrix whose elements are given in the Appendix.

9. APPLICATIONS

In this section, we provide two applications to real data to illustrate the flexibility of the TGW and TGBrX models presented in Section 4. The goodness-of-fit statistics for these models are compared with other competitive models and the MLEs of the model parameters are determined.

Data Set I: The Nicotine Data

The first data set refers to nicotine measurements, made from several brands of cigarettes in 1998, collected by the Federal Trade Commission. The report entitled tar, nicotine, and carbon monoxide of the smoke of 1206 varieties of domestic cigarettes for the year of 1998 consists of the data sets and some information about the source of the data, smokers behavior and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. This data set consists of n = 346 observations. These data have been used by Afify et al. (2016) to fit the Marshall-Olkin additive Weibull distribution

We shall compare the fits of the TGW distribution with those of other competitive models, namely: the Kumaraswamy-transmuted exponentiated modified Weibull distribution (Kw-TEMW) (Al-Babtain et al., 2015), transmuted exponentiated modified Weibull (TEMW) (Eltehiwy and Ashour, 2013), transmuted additive Weibull (TAW) (Elbatal and Aryal, 2013), Kumaraswamy modified Weibull (Kw-MW) (Cordeiro et al., 2014), beta Weibull (BW) (Lee et al., 2007), Kumaraswamy Weibull (Kw-W) (Cordeiro et al., 2010), and additive Weibull (AW) (Xie and Lai, 1995) distributions with corresponding densities (for $x > 0$):

Kw-TEMW: $$f(x) = ab\delta e^{-(\alpha x+\gamma x^{\beta})}(\alpha + \gamma\beta x^{\beta-1})\left[1-e^{-(\alpha x+\gamma x^{\beta})}\right]^{a\delta-1}$$
$$\times\left\{1+\lambda-2\lambda\left[1-e^{-(\alpha x+\gamma x^{\beta})}\right]^{\delta}\right\}\left\{1+\lambda-\lambda\left[1-e^{-(\alpha x+\gamma x^{\beta})}\right]^{\delta}\right\}^{a-1};$$

TEMW:

$$f(x)=\delta\left[\alpha+\gamma\beta x^{\beta-1}\right]e^{-\left(\alpha x+\gamma x^{\beta}\right)}\left\{1-e^{-\left(\alpha x+\gamma x^{\beta}\right)}\right\}^{\delta-1}\left\{1+\lambda-2\lambda\left[1-e^{-\left(\alpha x+\gamma x^{\beta}\right)}\right]^{\delta}\right\};$$

TAW: $$f(x)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\left\{1-\lambda+2\lambda e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right\};$$

Kw-MW:

$$f(x)=ab\gamma\left(\beta+\alpha x\right)x^{\beta-1}e^{\left(\alpha x-\gamma x^{\beta}e^{\alpha x}\right)}\left[1-e^{-\left(\gamma x^{\beta}e^{\alpha x}\right)}\right]^{a-1}\left\{1-\left[1-e^{-\left(\gamma x^{\beta}e^{\alpha x}\right)}\right]^{a}\right\}^{b-1};$$

BW: $$f(x)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}e^{-b(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1};$$

Kw-W: $$f(x)=ab\beta\alpha^{\beta}x^{\beta-1}\,e^{-(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}\left\{1-\left[1-e^{-(\alpha x)^{\beta}}\right]^{a}\right\}^{b-1};$$

AW: $$f(x)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}.$$

The parameters of the above densities are all positive real numbers except the parameter λ where $|\lambda|\leq 1$.

Data Set II: The Gauge Lengths Data

The second data set (gauge lengths of 20 mm) (Kundu and Raqab, 2009) consists of 74 observations. This data set is previously studied by Afify et al. (2016) to fit the Kumaraswamy complementary Weibull geometric distribution. For this data set, we shall compare the fits of the TGBrX distribution with those of other competitive models, namely: the generalized transmuted Burr X (GT-BrX) (Nofal et al., 2015), McDonald Weibull (Mc-W) (Cordeiro et al., 2014), modified beta Weibull (MBW) (Khan, 2015), exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al., 2015), T-BrX and BrX models with corresponding densities (for $x>0$):

GT-BrX: $$f(x)=2\alpha\beta^{2}xe^{-(\beta x)^{2}}\left[1-e^{-(\beta x)^{2}}\right]^{\alpha a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-e^{-(\beta x)^{2}}\right]^{\alpha b}\right\};$$

Mc-W: $$f(x)=\frac{\beta c\alpha^{\beta}}{B(a/c,b)}\,x^{\beta-1}\,e^{-(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}\left\{1-\left[1-e^{-(\alpha x)^{\beta}}\right]^{c}\right\}^{b-1};$$

MBW: $$f(x)=\frac{\beta\alpha^{-\beta}c^{a}}{B(a,b)}\,x^{\beta-1}\,e^{-b\left(\frac{x}{\alpha}\right)^{\beta}}\left[1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a-1}\left\{1-(1-c)\left[1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]\right\}^{-a-b};$$

$$\text{ETGR: } f(x) = 2\alpha\delta\beta^2 \; x \, e^{-(\beta x)^2}\left[1-e^{-(\beta x)^2}\right]^{\alpha\delta-1}\left\{1+\lambda-2\lambda\left[1-e^{-(\beta x)^2}\right]^{\alpha}\right\}$$
$$\times\left\{1+\lambda-\lambda\left[1-e^{-(\beta x)^2}\right]^{\alpha}\right\}^{\delta-1}.$$

The parameters of the above densities are all positive real numbers except the parameter λ where $|\lambda| \leq 1$.

In order to compare the fitted models, we consider some goodness-of-fit criteria like the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$), consistent Akaike information criterion ($CAIC$), $-2\hat{\ell}$, where $\hat{\ell}$ is the maximized log-likelihood, Anderson-Darling (A^*) and the Cramér-von Mises (W^*) statistics. The better distribution corresponds to smaller AIC, BIC, $HQIC$, $CAIC$, A^* and W^* values.

Table 1
Goodness-of-Fit Statistics for Data Set I

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*
TGW	212.176	220.176	235.562	226.303	220.293	0.34466	1.89074
Kw-TEMW	215.674	229.674	256.599	240.396	230.005	0.37863	2.08814
TEMW	215.967	225.967	245.199	233.625	226.143	0.38319	2.14169
TAW	217.393	227.393	246.625	235.051	227.569	0.37208	2.08766
Kw-MW	221.938	231.938	251.17	239.596	232.114	0.43426	2.52687
BW	225.173	233.173	248.559	239.3	233.29	0.49664	2.89774
Kw-W	226.184	234.184	249.57	240.311	234.302	0.5325	3.08454
AW	226.581	234.581	249.966	240.707	234.698	0.55222	3.17512

Table 2
Goodness-of-Fit Statistics for Data Set II

Model	$-2\hat{\ell}$	AIC	CAIC	HQIC	BIC	W^*	A^*
TGBrX	104.316	112.316	112.896	115.993	121.533	0.03531	0.25151
GT-BrX	108.055	118.055	118.937	122.65	129.575	0.10458	0.68807
Mc-W	108.784	118.784	119.667	123.38	130.305	0.1196	0.77957
MBW	109.145	119.145	120.028	123.741	130.666	0.12414	0.81141
ETGR	113.4	121.352	121.9	125.029	130.6	0.20714	1.3407
T-BrX	123.61	129.61	129.95	132.376	136.5	0.16923	1.28629
BrX	135.202	139.202	139.371	141.041	143.811	0.13403	0.86836

Table 3
MLEs and their Standard Errors for Data Set I

Model	Estimates (Standard Errors)				
TGW	$\alpha = 2.1296(0.67)$	$\beta = 1.523(0.295)$	$\theta = 0.1413(0.142)$	$\lambda = -0.4468(0.332)$	
BW	$\alpha = 0.6686(0.578)$	$\beta = 3.1645(0.426)$	$a = 0.7784(0.163)$	$b = 3.0922(8.174)$	
Kw-W	$\alpha = 0.6157(0.392)$	$\beta = 3.1187(0.698)$	$a = 0.8395(0.233)$	$b = 3.7931(6.921)$	
AW	$\alpha = 1.135(0.062)$	$\beta = 0.3084(0.1)$	$\gamma = 0.0002(0.001369)$	$\theta = 2.7219(0.114)$	
TEMW	$\alpha = 0.6977(0.492)$	$\beta = 2.5908(0.265)$	$\gamma = 1.1925(0.259)$	$\delta = 1.5007(0.487)$	$\lambda = -0.6328(0.228)$
TAW	$\alpha = 1.2252(0.239)$	$\beta = 0.8994(0.091)$	$\gamma = 0.433(0.229)$	$\theta = 2.6404(0.267)$	$\lambda = -0.8831(0.147)$
Kw-MW	$\alpha = 0.6145(0.09)$	$\beta = 0.4466(0.364)$	$\gamma = 0.5622(0.353)$	$a = 4.3285(3.595)$	$b = 6.7039(6.728)$
Kw-TEMW	$\alpha = 0.113(0.22)$ $a = 0.47(0.213)$	$\beta = 2.316(0.62)$ $b = 1.079(1.828)$	$\gamma = 1.436(1.71)$	$\delta = 2.033(1.145)$	$\lambda = -0.902(0.197)$

Table 4
MLEs and their Standard Errors for Data Set II

Model	Estimates (Standard Errors)				
GTBrX	$\alpha = 3.4900(2.084)$	$\beta = 0.6615(0.120)$	$\lambda = 0.0019(0.048)$	$a = 2.5190(1.503)$	$b = 0.0161(0.428)$
Mc-W	$\alpha = 1.4383(1.447)$	$\beta = 0.5832(0.211)$	$a = 83.7204(78.89)$	$b = 14.4281(15.87)$	$c = 3.4606(9.663)$
MBW	$\alpha = 1.7656(1.097)$	$\beta = 1.4265(1.488)$	$a = 36.3366(4.439)$	$b = 3.3618(6.695)$	$c = 3.0967(4.714)$
TG-BrX	$\alpha = 0.7477(0.891)$	$\beta = 0.8516(0.011)$	$\lambda = -0.1444(0.856)$	$\theta = 0.0092(0.018)$	
ETGR	$\alpha = 2.1214(0.315)$	$\beta = 0.6985(0.040)$	$\lambda = 0.3201(0.228)$	$\delta = 7.790(1.727)$	
T-BrX	$\alpha = 5.5052(0.776)$	$\beta = 0.6245(0.017)$	$\lambda = 0.3599(0.253)$		
BrX	$\alpha = 7.784(1.625)$	$\beta = 0.6445(0.024)$			

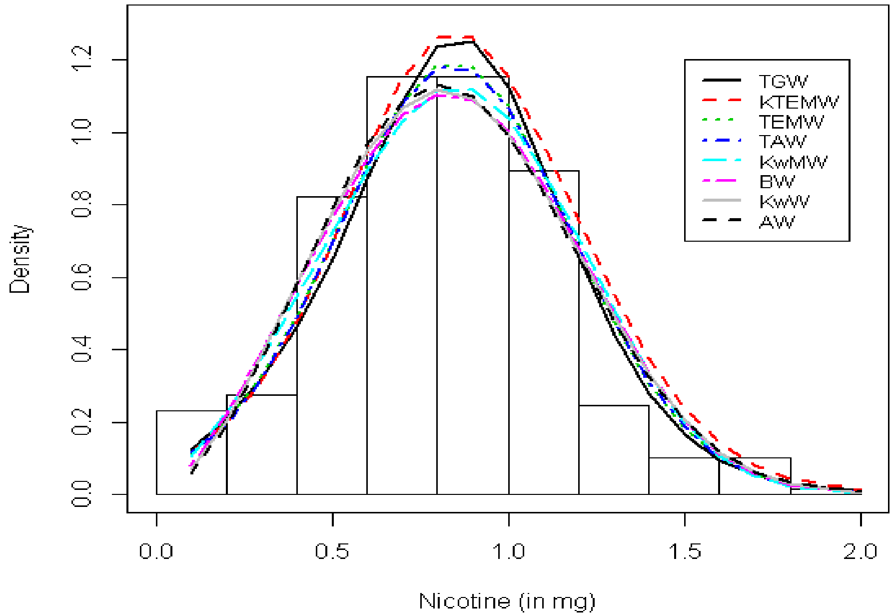

Figure 3: Fitted pdf of TGW Model and other Distributions for Data Set I

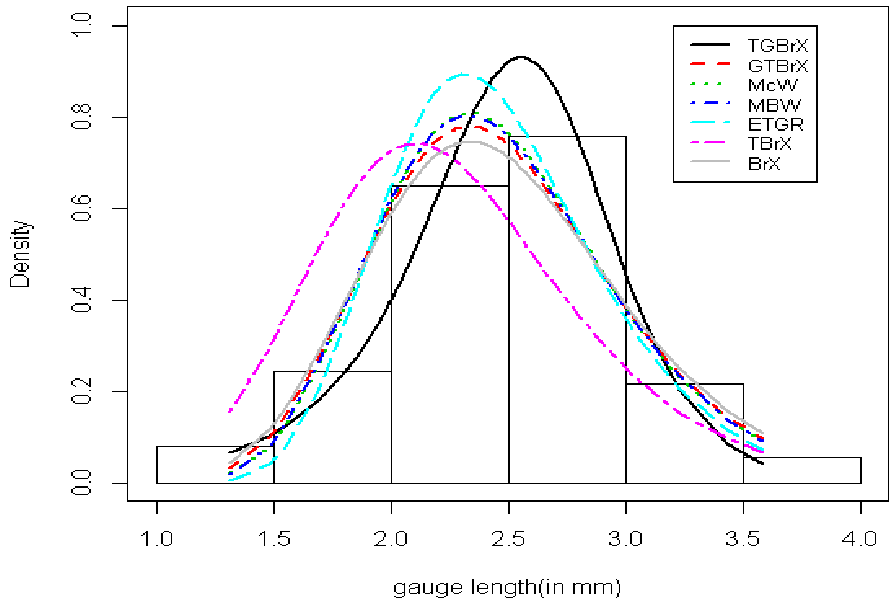

Figure 4: Fitted pdf of TGBrX Model and other Distributions for Data Set II

Tables 1 and 2 list the values of goodness-of-fit statistics whereas the MLEs of the model parameters and their standard errors are given in Tables 3 and 4. The histogram of the nicotine data and the estimated densities are displayed in Figure 3. Figure 4, displays the histogram of the gauge lengths data and the estimated densitie.

In Table 1, we compare the fits of the TGW model with the Kw-TEMW, TEMW, TAW, Kw-MW, BW, Kw-W and AW distributions. The values in these tables indicate that the TGW model has the lowest goodness-of-fit statistics (for data set I) among the fitted models. So, the TGW model could be chosen as the best model for the subject data.

Similarly, in Table 2, we compare the fits of the TGBrX model with the GT-BrX, Mc-W, MBW, ETGR, T-BrX and BrX distributions. It is shown that the TGBrX model has the lowest goodness-of-fit statistics values (for data set II) among all fitted models. So, the TGBrX model can be chosen as the best model for the subject data. It is clear from Tables 1 and 2 and Figures 3 and 4 that these special case of TG-G family provide the best fit to both data sets.

10. CONCLUSIONS

There is a great interest among statisticians and practitioners in the past decade to generate new extended families from classic ones. We present a new *transmuted geometric*-G (TG-G) family of distributions, which extends the transmuted family by adding one extra shape parameter. The mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations, entropies, order statistics and probability weighted moments are provided. The model parameters are estimated by the maximum likelihood estimation method and the observed information matrix is determined. It is shown, by means of two real data sets, that special cases of the TG-G class can give a better fit than other models generated by well-known families.

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APPENDIX

The elements of the observed information matrix are:

$$U_{\lambda\lambda}=-\sum_{i=0}^{n}\frac{1}{p_i^2}\left[1-\frac{2\theta G(x;\phi)}{s_i}\right]^2,$$

$$U_{\lambda\theta}=-2\sum_{i=0}^{n}\frac{G(x;\phi)\left[s_i-\theta G(x;\phi)\right]}{p_i s_i^2}+\sum_{i=0}^{n}\frac{\left[s_i-2\theta G(x;\phi)\right]\left[2\lambda G(x;\phi)s_i-2\lambda\theta G(x;\phi)^2\right]}{p_i^2 s_i^3},$$

$$U_{\lambda\phi}=-4\sum_{i=0}^{n}2\frac{G'(x_i;\phi)}{p_i s_i}+\sum_{i=0}^{n}\frac{G(x;\phi)\left[(1-\theta)G'(x_i;\phi)\right]}{p_i s_i^2}+\sum_{i=0}^{n}\frac{\left\{2\lambda s_i^{-1}G'(x_i;\phi)-2\lambda s_i^{-2}G(x;\phi)\left[(1-\theta)G'(x_i;\phi)\right]\right\}}{p_i^2 s_i\left[s_i-2G(x;\phi)\right]^{-1}},$$

$$U_{\theta\theta}=\frac{-n}{\theta^2}-2\sum_{i=0}^{n}\frac{s_iG'(x;\phi)-G(x;\phi)^2}{s_i^2}-2\lambda\sum_{i=0}^{n}\frac{G(x;\phi)^2(s_i-1)}{p_i s_i^2}+\sum_{i=0}^{n}\frac{\left[2p_is_iG(x;\phi)-2\lambda G(x;\phi)s_i-2\lambda\theta G(x;\phi)^2\right]}{\left(p_is_i^2\right)^2\left[2\lambda G(x;\phi)s_i-2\lambda\theta G(x;\phi)^2\right]^{-1}},$$

$$U_{\theta\phi}=-2\sum_{i=0}^{n}\frac{G'(x_i;\phi)\left[s_i-(\theta-1)G(x;\phi)\right]}{s_i^2}-2\lambda\sum_{i=0}^{n}\frac{G'(x_i;\phi)\left[(\theta-1)G(x;\phi)+s_i-2\theta G(x;\phi)\right]}{p_i s_i^2}+4\lambda\sum_{i=0}^{n}\frac{G'(x_i;\phi)\left\{(\theta-1)p_is_i-\lambda\theta s_i-\lambda\theta(\theta-1)G(x;\phi)\right\}}{\left(p_is_i^2\right)^2\left[G(x;\phi)s_i-\theta G(x;\phi)^2\right]^{-1}}$$

and

$$U_{\phi_k\phi_k} = \sum_{i=0}^{n} \frac{g(x;\phi)\, g''(x_i;\phi) - \left[g'(x_i;\phi)\right]^2}{g(x;\phi)^2} - 2\lambda\theta \sum_{i=0}^{n} \frac{(\theta-1)\left[G'(x_i;\phi)\right]^2 + s_i G''(x_i;\phi)}{p_i s_i^2}$$
$$+4\lambda\theta \sum_{i=0}^{n} \frac{\left[G'(x_i;\phi)\right]^2 s_i \left\{(\theta-1)\, p_i s_i - \lambda\theta s_i + \lambda\theta(\theta-1)G(x;\phi)\right\}}{p_i s_i^2}$$
$$+2\lambda\theta(\theta-1) \sum_{i=0}^{n} \frac{G(x;\phi)\, G''(x_i;\phi) + \left[G'(x_i;\phi)\right]^2}{p_i s_i^2}$$
$$-4\lambda\theta(\theta-1) \sum_{i=0}^{n} \frac{G(x;\phi)\left[G'(x_i;\phi)\right]^2 \left\{(\theta-1)\, p_i s_i - \lambda\theta s_i + \lambda\theta G(x;\phi)(\theta-1)\right\}}{\left(p_i s_i^2\right)^2},$$

where $g''(x_i;\phi) = \partial^2 g(x_i;\phi)/\partial\phi_k^2$ and $G''(x_i;\phi) = \partial^2 G(x_i;\phi)/\partial\phi_k^2$.